Research-Based Guidelines for Selecting Mathematics Curriculum

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Why Another Set of Adoption Guidelines?

In the early 1900's, the challenge of integrating thinking and content area knowledge was formidable, even in the education of the elite. Teachers today face much greater challenges. "Although it is not new to include thinking, problem solving, and reasoning in *someone's* school curriculum, it is new to include it in *everyone's* curriculum" (Resnick, p. 7).

Additionally, regular classroom teachers face even greater challenges as a result of the Regular Education Initiative (REI), the movement that results in the placement of increasing numbers of special education students into regular classrooms.

The purpose of these adoption guidelines is to provide assistance for teachers, through the adoption process, in their quest to meet exceptionally difficult challenges with respect to their mathematics programs:

1. Effectively teach this extremely diverse group of students. Regular classrooms were quite diverse before REI, and have become more diverse since. Because of the multitude of ways students are referred for special education services, it has been estimated that up to 80% of American school children could qualify for special services in mathematics somewhere. That is to say, a teacher is likely to have several "regular" students in the room who are having difficulty with mathematics, while at another school, these same students would qualify for special services using a different scheme for referral.
2. Teach important mathematics. Rightfully, mathematics educators are calling for a much greater emphasis upon the mathematics that students are most likely to find useful in their lives.

These guidelines have been developed specifically as a means of addressing these two challenges effectively: teaching important mathematics to widely diverse learners. Consequently, they should be thought of as supplemental guidelines. They are modest guidelines, in the sense that they do not address the kinds of general criteria commonly listed in adoption guidelines: lack of sexual and other bias, attractiveness, and so on. However, there is little doubt that the instructional focus of these guidelines is becoming increasingly critical to teachers.

Figure 1 illustrates the nature of the new challenges. Traditionally, the curriculum in regular classrooms with no mainstreaming focused upon a relatively narrow "middle" section of students: all but the very lowest and the gifted. The lowest, "at-risk" students would eventually be retained, or would struggle along, or would be referred for special education or other specialized, basic services. Gifted students, similarly, might be referred to special programs for the gifted, or be accommodated by enrichment activities designed by teachers or provided as supplements to instructional materials.

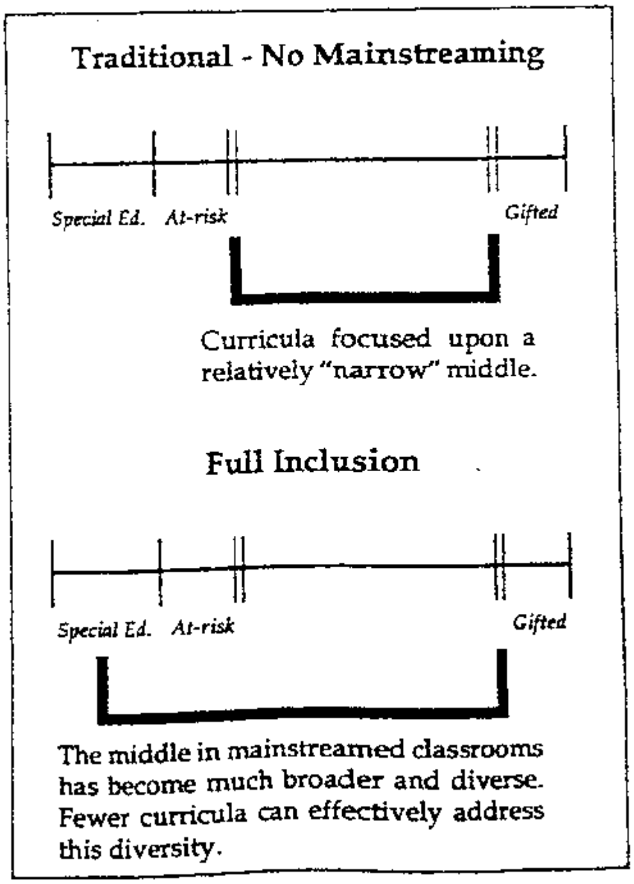


Figure 1. Curricula for Diverse Learners

With mainstreaming (and with "full inclusion" models especially), the diversity in a regular classroom can be staggering. Both the number and diversity of the "middle" students in such classrooms has increased. Imagine, for example, a classroom comprising twenty "regular" students and eight special education students. Imagine further that *all* students are mainstreamed. Of the eight special students, perhaps two have handicapping conditions such that they will require attention from a specialist in the classroom. Of the twenty regular students, a fourth or more may have difficulty with mathematics: they are behind the other fifteen students, and are at risk of eventual classification as special education students. One or two of those twenty regular students may be gifted. If the classroom teacher has primary responsibility for the fifteen regular students who are average or above, and a specialist has primary responsibility for the two most severely handicapped students, who has responsibility for the eleven "low" students? How many different mathematics lessons can one reasonably expect the teacher and/or the specialist to deliver each day? A traditional curriculum that targets the low-average to high range is pitifully inadequate for today's diverse regular classrooms.

> *Materials selected according to the criteria outlined in these guidelines can go a long way toward helping to make mainstreamed classrooms more manageable*

Such difficulties, of course, cannot be entirely resolved through the judicious selection of instructional materials. However, materials selected according to the criteria outlined in these guidelines can go a long way toward helping to make mainstreamed classrooms more manageable. The few criteria focused upon in these guidelines are those shown by currently available research to promise the greatest acquisition of mathematics knowledge by the broadest range of diverse learners. An instructional program based upon these criteria cannot be expected to be the ideal curriculum for two groups of students within a diverse classroom: the most severely handicapped, and the very highest performing students.

Returning to the classroom scenario above, the guidelines are intended to help adoption committees select materials that will be effective with the six "non-severe" special education students, the five low regular students, and perhaps all but the five highest of the remaining fifteen regular students. The regular teacher could teach mathematics to those twenty-one students using such materials, while the specialist continues to have primary responsibility for the severely handicapped students. And what of the five highest students? As you will see, these guidelines are not likely to restrain the mathematics growth of higher-performing students. As a consequence, those five could participate with the other twenty-one in the regular mathematics lesson, so long as that work is supplemented (by either the specialist or the teacher) with more challenging material.

The focus of these guidelines, then, is upon all but the very lowest students (who require the services of specialists in any circumstance) and the very highest (who are the most able to work independently, and to learn well from indirect, discovery-oriented activities).

Although circumstances vary considerably, and scenarios like the one above are not perfect solutions to complex problems, accommodating the greatest possible diversity through the low to high-average range promises greater manageability in classrooms with mainstreamed disabled learners.

The Organization and Use of These Guidelines

There are six individual guidelines presented herein. Each is summarized on one of six single-page charts. Essentially, those charts *are* the adoption guidelines. That is, we cannot realistically expect every member of an adoption committee to read the document you are reading now. Rather, we anticipate that perhaps one member of a committee will read this document: a committee chairperson, or someone assigned especially to investigate the needs of mainstreamed learners. In that sense, this document is a backup to the six guideline charts. The charts, and this document as well, are backed up further by a technical report (Dixon, 1992).

As a practical matter, each member of an adoption committee needs only the six guideline charts in order to evaluate the extent to which various instructional materials accommodate diverse learners. Although guidelines may be applied to materials in any order, we suggest:

1. Big Ideas
2. Conspicuous Strategies
3. Mediated Scaffolding
4. Strategic Integration
5. Primed Background Knowledge
6. Review

Generally, this is the order of importance. For example, #1 - Big Ideas, refers to teaching important mathematical concepts. Other guidelines, such as review, are not likely to impact much on understanding if a program focuses attention on unimportant, or even trivial concepts.

The Guidelines

Big Ideas

In order for students to solve interesting, complex, realistic problems, they must acquire knowledge of the more important, key mathematical concepts: concepts recently referred to in mathematics literature as "Big Ideas." Big ideas within a content area are those concepts, principles, or heuristics that facilitate the greatest amount of knowledge acquisition and understanding across the rest of that content area. That is, big ideas make it possible for students to learn the most and learn it well, as efficiently as possible. Consider, for example, the following problem:

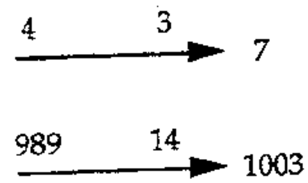
> At lunch, each student can choose a carton of white or chocolate milk. Each fifth-grade class is to estimate how many cartons of chocolate and white milk should be ordered for the entire school.

In addition to computational ability, for students to work such problems successfully, they must fully understand the big mathematical ideas of proportions and estimation, concepts that recur frequently in a broad range of real mathematical problems.

An all too common alternative to big ideas is broad *coverage* of or *exposure* to numerous objectives. Neither the term coverage nor exposure implies the kind of learning with understanding to which educators generally aspire for regular classroom and mainstreamed learners alike.

Several "small" mathematical ideas can often be best understood in relationship to a larger, "umbrella concept." For instance, if the geometric concept of area is taught to a high level of understanding, then students need not learn formulas by rote for calculating area. In turn, the understanding of area provides the underpinnings of another important mathematical concept: volume, as a function of base (area) times height. Approached this way, the seven traditional formulas for computing volume can be reduced to a single formula (base times height) with two slight variations.

Finally, even something as "simple" as math facts can be taught in terms of larger mathematical concepts, rather than as a large set of discrete and unrelated facts to be recalled by rote. For instance, each addition fact bears an "adding on" relationship to some other fact: 4 + 4 = 8, and 4 + 5 is one more than 4 + 4, so the sum is also one more, 9. A more sophisticated approach to facts is to capitalize upon the inherent relationships among addition and subtraction facts alike. For example, all four of these problems: 3 + 4, 4 + 3, 7 - 4, and 7 - 3 are all part of a single "family" of relationships. Similarly, "larger" problems also bear family relationships to one another: 1003 - 989, 989 + 14, etc. Such interrelationships can be mapped:



The mapping strategy can also be expanded beyond facts to solve verbal math problems.

Evaluating Big Ideas. Although authorities in mathematics education are increasingly emphasizing the importance of teaching big ideas, they are yet to develop a definitive list of important mathematical concepts. If there were such a list adoption committees could simply check off program concepts against a list. Still, several concepts seem to be generally accepted as crucial, big ideas: place value; the addition, subtraction, multiplication, and division of whole numbers; fractions, including ratios and proportions; estimation and approximation; probability; volume and area; and all of these concepts incorporated into verbal problem-solving.

In an instructional program that emphasizes big ideas, a given idea is likely to appear repeatedly, in different contexts. For instance, the "identity principle" in mathematics says that the value of a number does not change when multiplied by one, *in any form* (such as 23/23, 4ay/4ay, etc.). Such an idea is big (key, important) because it facilitates understanding of so many aspects of mathematics, ranging across fractions, ratios, and algebra. Thus, one step in evaluating an instructional program for big ideas is to search for a major mathematical concept that is applied in a variety of contexts.

Another step in evaluating big ideas is effective, yet relatively simple. Significantly more time will be allocated to big ideas than to minor ideas. At a given level of an instructional program, a single big idea and a relatively small idea can be compared upon the basis of the time allocated to teaching (or covering or exposing) each. If, for example, all outcomes in a program are pursued in units or chapters of equal length, that program *does not* differentiate big ideas from others. In contrast, if certain ideas occur repeatedly in a program and more time is allocated to those ideas than to several others, then big ideas are probably being accommodated.

The highest performing students are those most likely to, eventually, infer useful strategies from their experience. Explicitly taught strategies, in effect, let the majority of students in on the "secrets" to success.

Conspicuous Strategies

Someone who easily and fluently solves new mathematical problems uses some kind of "strategic approach" to each new problem-solving situation: some kind of expert strategy. In many cases, experts cannot even clearly articulate the details of their strategies. The purpose of strategy instruction is to clearly present learners with strategies like those that experts use. The highest performing students are those most likely to, eventually, infer useful strategies from their experience. Explicitly taught strategies, in effect, let the majority of students in on the "secrets" to success.

A strategy, then, is a somewhat general set of steps students follow in solving problems. Strategies may be so narrow that they result in rote-like performance on a very limited set of problems. In contrast, a strategy can be so broad that it doesn't work for the majority of students, the majority of time. Simply "drawing a picture," for example, can in fact help some students solve some problems, but is too broad to qualify as a reliable strategy for most students.

"Medium" strategies are the most likely to benefit students. Consider this problem:

It takes the attendance office two minutes to process three tardy students. How long will it take them to process eleven tardy students?

A "medium" strategy is to first map the "units" in such a problem:

$$\frac{\text{minutes}}{\text{tardy students}}$$

Next, write the known quantities with the units:

minutes	2	
tardy students	3	11

Finally, map the missing quantity:

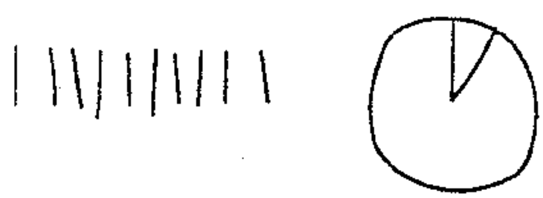
$$\frac{\text{minutes}}{\text{tardy students}} \quad \frac{2}{3} = \frac{\square}{11}$$

All that remains is the computation necessary to solve this correctly mapped problem.

Evaluating Strategies. First, determine whether a strategy is taught explicitly at all. Only a limited number of students are likely to infer effective strategies, and the process of inferring strategies is very time-consuming (inefficient) in any case.

Next, make a preliminary determination of how narrow or broad strategies appear to be. Can the strategy be applied to a large number of problems? Can it be applied reliably?

Determining the "reliability" of a strategy can be a challenging and engaging activity. A good technique is to role-play a student, one that is on the lower end of the targeted student population, and perhaps one that is a bit contrary. In that role, the objective is to attempt to "break the strategy" by *identifying any instance in which one can follow the strategy, but still not solve the problem.* As a simple example, a student could easily "draw a picture or map" of the tardy student problem above, but without indicating the relationships necessary to solve the problem. Here is one such student drawing:



Playing the role of the student can take a little practice. It requires one to focus strictly upon the information given in strategies and problems and not use one's own knowledge. An approach that can be effective is for someone on the adoption committee to locate problem types in the upper levels of the material under consideration that the committee person has genuinely had a difficult time understanding for himself or herself in the past. If the strategy does not facilitate understanding for such an educated adult, it certainly is unlikely to facilitate understanding for students. It is usually safe to generalize the findings from evaluating the upper levels of materials to the lower levels.

Mediated Scaffolding

British educator A. J. Romiszowski has characterized traditional mathematics instruction as: "I'll work two on the board, then you do the rest." The "I'll work two" part of that approach can be thought of as a model, and the "you do the rest" is considered immediate testing. It has been said that the problem with learning from experience is that the lessons come too late. The same could be said of this traditional model of instruction. After "doing the rest," students might receive feedback—ranging from right/wrong to an explanation of how to do missed problems—and possibly a grade. The feedback is too late, and the grade, too early.

Scaffolding is a means by which students receive support in various forms along the path to full understanding and "doing the rest" successfully. An analogy helps illustrate the potential power of scaffolding in mathematics instruction. Imagine trying to teach a physically disabled youngster or a very young child how to slide down a playground slide. We might begin by carrying the child up the steps and holding her in our lap as we slide. That first phase of teaching illustrates a great deal of support—scaffolding.

After proceeding this way a few times, we might remove a bit of the scaffolding by, perhaps, allowing the child to take one or two steps of the ladder with minimal help, or by placing the child low on the slide and allowing her to slide on her own a very short distance. And so on.

All along, we would attempt to remove more bits of scaffolding, but in no instance would we abruptly remove all the scaffolding and, in essence say, "you do the rest." Obviously, doing so could result in serious physical injury. Analogously, removing scaffolding prematurely—or worse, never providing any—can result in serious intellectual injury for many students. Such injury is less obvious than physical injury, but just as real, just as likely, and perhaps over time, more impairing. After all, physical injuries heal. Many mathematical injuries are extremely difficult to rehabilitate.

Teachers and peers can provide scaffolding, independently of instructional materials. However, materials can and should provide scaffolded sequences of tasks for students to perform. Highly scaffolded tasks may be quite contrived and "look strange," just as carrying a child in the early steps of "sliding" does not look much like "real sliding." The crucial thing is that tasks clearly and systematically provide gradually reduced scaffolding so that ultimately, students solve mathematical problems effortlessly, and with few injuries along the way.

Evaluating Scaffolding. Select an important strategy from a grade level, preferably a strategy on a concept already identified as a big idea. Focus examination on all the tasks associated with the selected concept, from the very beginning, to the point at which students are expected to apply their knowledge independently.

The first tasks should be designed to ensure that even relatively low students can perform them successfully. Gradual—perhaps even subtle—changes in tasks should be discernible, changes that reflect a shift to greater student understanding and independence.

Strategic Integration

The principal benefit to all learners of integrating knowledge is that it enhances the likelihood that students will learn *when to apply* their knowledge. There are many concepts in mathematics that are similar to others, yet significantly different. For instance, from the point of view of someone just learning fractions, an addition problem and a multiplication problem appear to be very similar to one another, yet very different knowledge is required to solve each:

$3/4 + 4/5 = ?$ $\qquad$ $3/4 \times 4/5 = ?$

The benefit of integrating is probably most apparent when compared with the difficulties students encounter with "non-integrated" knowledge. They may, for example, simply add the numerators and denominators of addition fraction problems such as the one above, in spite of the fact that when adding fractions was first being taught, students seemed to "get it." Perhaps they had little trouble with adding fractions initially, when addends had like denominators. Perhaps they subsequently caught on to adding fractions with unlike denominators. But after teaching multiplication of fractions, without integration, confusion developed: Which problem type requires like denominators? When does one operate on both the numerator and the denominator?

The predictable outcome of not integrating knowledge is an abundance of stubborn misconceptions: using area formulas to compute volume, subtracting a big number from a small number in whole number subtraction, difficulties with fractions with values greater than one, and so on. Although such misconceptions are related to the quality of initial teaching (big ideas, strategies), even topics that are well understood initially can result in confusion without integration.

The primary means of integrating knowledge is quite simple, perhaps elusively simple: just do it! That is, the developers of instructional materials can considerably advance the goals of fully integrated knowledge by "putting things together." In the "real world of mathematics," we are constantly called upon to pull some specific knowledge from our entire base of mathematics knowledge and use that specific knowledge appropriately. Instructional materials should anticipate that, in addition to possessing knowledge, students need to learn *when* to use components of that knowledge.

Evaluating Integration. Select any common mathematics misconception which committee members have observed in students. Identify specifically what gets confused with what in that misconception. For example, if students add the numerators and denominators of fractions, they are probably confusing addition with multiplication. Examine the materials under consideration to determine the extent to which the potentially confusing concepts are mixed with one another.

Initial teaching examples can also be used to facilitate integrated knowledge. If all the examples students work with as they initially learn fractions have a value of less than one, then it is quite predictable that many students will draw the reasonable—but wrong—conclusion that all fractions are "small pieces of things," with values of less than one. Look at initial sets of teaching examples to determine whether some inadvertent misconceptions are actually implied by the examples used.

New problem types can be related to familiar problems taught earlier. Students can be shown specifically why Strategy "A" is not adequate for solving problems in "B" situations. Look for examples such as: "This looks like a problem that can be solved with _______ procedure, but that won't work. Here's why: ___________." An example of one concept or strategy is always a "negative example" of another. Full understanding requires that student know not only what something is, but also, what it is not.

Primed Background Knowledge

It nearly goes without saying that learning most mathematics, and especially most important mathematics, is dependent upon some knowledge the learner already possesses. This fact is generally accommodated in most instructional materials, but not always effectively.

Three different practices are of limited value. The first is teaching prerequisite knowledge through "coverage" or "exposure." If prerequisite material was just covered, or students were otherwise exposed to it, it is not likely to have been learned well enough to facilitate the learning of new, complex knowledge.

Even when prerequisite knowledge is taught thoroughly, the timing of that instruction in relation to the new knowledge can be crucial. The second and third practices that are inadequate relate to timing. On the one extreme, the prerequisite knowledge may have been taught far in advance of the instruction on the new knowledge. In that case, many students are likely to lose their facility with the prerequisite knowledge, if not to forget it altogether. At the other extreme, prerequisite knowledge is sometimes taught in too close proximity to the new knowledge, often on the same day, during the same instructional lesson. In this case, students have no time to truly learn, assimilate, and understand the prerequisite knowledge before the new knowledge is introduced.

Ideally, prerequisite knowledge is introduced (or reviewed) at a given point, and used for a period of a few days or weeks before the introduction of the new, more complex knowledge. Such a schedule means that students have the opportunity to develop sufficient fluency for using the prerequisite knowledge, without the opportunity of having that fluency diminish or disappear as a result of disuse.

Evaluating Prior Knowledge. To evaluate material based upon this guideline, it is necessary to locate one or more complex topics in a given level of the instructional program under consideration. Next, that topic must be analyzed to determine what prerequisite knowledge is assumed for learners. First, determine whether the prerequisite knowledge has been taught (or reviewed) thoroughly (rather than simply covered or "taught for exposure").

Next, note the time interval between the instruction on the prerequisite knowledge and the new knowledge. A "medium" time interval (of a few days or weeks of nearly continuous, if light, practice) is desirable. Neither long intervals without intervening practice nor extremely short intervals are likely to be effective.

Review

The term "review" can be an emotive one in education, conjuring up images of endless (and, perhaps, mindless) drill and practice. Yet research strongly supports certain review practices as significantly effective. We include review as the last guideline because in many ways, effective review is dependent upon the extent to which other guidelines are implemented in instructional materials. It can be said that one gets out of review what one puts into it. That is, the quality of instruction—principally in terms of big ideas and strategies—influences the value of review. Regardless of how "small ideas" or marginally significant material is reviewed, the ideas remain small, the material marginally significant.

The following are requirements for effective review:

1. **Sufficient.** Is there enough review to achieve the goals of fluency and understanding?
2. **Distributed.** Given a fixed number of review opportunities, that number will enhance learning better if it is distributed over time than if it is massed. Specifically, distributed review contributes to long-term retention and automaticity of knowledge.
3. **Cumulative.** This requirement is tied closely to the integration guideline. The notion of cumulative review means that material taught accumulates in review. After A and B are taught, for example, A and B are reviewed together.
4. **Varied.** With relatively few exceptions, the specific items that are reviewed should not be the same as the items used earlier in instruction. The reason for this is that varied items promote generalization and transference. However, items should not be so varied that they actually represent new knowledge.

Evaluating Review. It is impossible to give a simple formula for determining how much review is sufficient. Instructional materials, however, should err on the side of "too much" review. When students need less review than a program provides, teachers can simply cut back the assignment of review work. In contrast, a great deal of teacher time and effort would be required to create additional review if the amount provided in a program is insufficient.

When examining a program for the distribution of review, the general amount of review can be noted, too. Evaluate the distribution of review by creating a summary of the review for a major concept taught in the program under consideration. Only tasks that students complete independently (including test tasks) should be considered "review." The results of such an evaluation can be represented in many ways. Here is one example of visually representing well-distributed review:

	Lesson Number								
	23	24	25	26	27	28	29	30	31
Number of Review Tasks	7	5		4		4			6

Determine whether the review in a program is cumulative. Select concepts that are frequently confused by students (multiplying and adding fractions, decimal versus whole number place value, subtracting with regrouping and adding with regrouping, etc.). Are the confused concepts reviewed together?

When review is cumulative, some form of scaffolding should be provided initially. For example, if students are reviewing the addition and multiplication of fractions together for the first time, the task should prompt the students to carefully note the operation in each problem before beginning computation.

Finally, determine whether review items vary appropriately. Assume, for example, that algorithms have been taught for subtracting whole numbers with no renaming required, and problems requiring renaming but with no zeros in the minuend. Both types of problems can, literally, be infinitely varied by varying the individual digits from 1 to 9, and by varying the number of digits in both the minuend and the subtrahend. However, problems including zeros in the minuend would not be a variation of the algorithms taught, but rather, they would represent new knowledge.

Possibly, the most difficult area of mathematics to evaluate for appropriate variation is problem solving. Frequently, instructional programs include many "review" problems that cannot actually be solved by following strategies previously taught. There are few tidy listing of problem types in mathematics; basically, a problem type is defined by the strategy taught to solve that type. Thus, evaluators should actually work a variety of review problems, being careful to apply only strategies taught in the program under consideration. If other, untaught knowledge is required to solve some "review" problems, then the variation of review is too great.

Summary of Guidelines and Impact on Diverse Learners

In developing these guidelines, it has been crucial to not only identify principles of effective instruction for lower-performing students, but to focus upon those principles that are effective with many other learners in mainstreamed regular classrooms as well. We stated in the beginning that instructional materials incorporating these guidelines are likely to be appropriate for all but the lowest and highest students in a highly diverse regular classroom. Specifically, these guidelines impact diverse learners in the following ways:

1. **Big Ideas.** Because the notion of "big ideas" is roughly comparable to important ideas, knowledge, and concepts, this principle is appropriate for all students, without modification for different ability levels. The principle of big ideas is learner independent: mathematical concepts that are important for understanding are important for everyone.
2. **Conspicuous Strategies.** All students can benefit from explicitly taught strategies. That, however, is not to say that every student requires explicit strategies as much as any other. Gifted students, for example, can benefit from explicitly taught strategies, but require such explicit instruction to a lesser extent that their lower-performing peers.
3. **Mediated Scaffolding.** The amount of scaffolding for any given strategy appears to vary as a function of ability. Lower students need more scaffolding; higher students need less. Instructional programs should provide scaffolding that is adequate for the lower-performing students, giving teachers the option of cutting back on scaffolding without needing to develop more than a program provides.
4. **Strategic Integration.** Knowing when to use knowledge is important to understanding for all learners.
5. **Primed Background Knowledge.** In theory, all students need the knowledge that is prerequisite to acquiring new, complex knowledge. In practice, however, it is the lower-performing students who are most likely to display deficits in prior knowledge. Either they never learned such knowledge, or did not learn it thoroughly, or learned it fairly well, but forgot it.
6. **Review.** The principles of effective and efficient review are generally most crucial for lower-performing students. Specifically, higher-performing students are likely to require less review. However, integrated (cumulative) review is important to understanding for all students, and varied review is important to transfer and generalization, regardless of ability level. The consequences of inappropriately varied review are most likely to cause difficulties for the lower-performing students. As is the case with scaffolding, instructional materials should provide review sufficient for the lower-performing students.

Note: Research references have been largely omitted from this report for ease in reading. A complete technical report supporting these conclusions can be obtained from the National Center to Improve the Tools of Educators, 805 Lincoln, Eugene, OR 97402. Ph: 1-503-683-7543. Ask for the technical report entitled: "Mathematics Guidelines for Diverse Learners."

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Considerations For Adopting Mathematics Instructional Materials
Big Ideas

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In order to promote deep understanding and problem-solving, students should be taught major, important mathematics concepts. That knowledge will make learning "subordinate" concepts easier and more meaningful. Big ideas represent central ideas within a discipline. They have rich explanatory and predictive power. And perhaps foremost, Big Ideas apply to many common, everyday contexts and situations.

Example: Volume is frequently taught as seven similar and confusing formulas that many students attempt to simply memorize. An alternative is to teach the fundamental concept of volume as a function of bases times height. When students have mastered the concept of "base," they can calculate any volume with two minor variations on the basic formula, area of the base times height: B x h.

Rectangular prisms, wedges, cylinders:	B x h = v
Rectangular pyramids, triangular pyramids, cones	B x 1/3 h = v
Spheres:	B x 2/3 h = v

Rationale: Topics in mathematics are often treated as unrelated. Moreover, there has been a proliferation of objectives or topics in published materials over the past several years. As a consequence, many topics are just "covered" and relatively minor topics get as much attention as very important topics: "big ideas" in mathematics. In order to understand mathematics, students should thoroughly *learn* (as opposed to "cover") the most important mathematics concepts.

Implications for Materials Evaluation. Big Ideas can be incorporated into instructional materials in a variety of ways, implying that materials can be examined for different manners in which big ideas may be accommodated.

1. Significantly more instructional time is allocated to big ideas than is allocated to other topics. For instance, if topics are typically addressed in weekly units, big ideas might be taught as sections of two or more units. This is a "quantitative" judgment: Big Ideas are the most important ideas, so more time (and/or space) is allocated to them.

2. Topics can be analyzed for "cross topic" big ideas, important mathematical relationships that apply to a wide range of traditional mathematical topics. For example, number families (e.g., $\xrightarrow{4 \quad 2}$ 6) represent the relation between addition and subtraction. For instance, $\xrightarrow{4 \quad \square}$ 6 calls for subtraction, 6 - 4 = □). The use of number families provides a single basis for a higher order organization for facts, a mapping strategy for problem solving, a method for analyzing data tables, and a foundation for simple algebraic relationships, such as 4 + □ = 6, so 6 - 4 = □.

This is more of a "qualitative" judgment that requires evaluators first to identify those mathematics ideas they believe to be most central, then to examine all those topics to which each important idea is most likely to apply. For example, if the "identity principle" is considered a crucial mathematics concept, then one might reasonably expect to see it used conspicuously in teaching division of fractions for deep understanding.

Considerations For Adopting Mathematics Instructional Materials

Conspicuous Strategies

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A strategy is a series of steps that students follow to achieve some goal. Such steps are an approximation of the steps experts follow covertly (and, perhaps, unconsciously) while working toward similar goals. In instruction, such steps are initially made overt and explicit for students. Eventually, as students master a strategy, the steps become more covert, as for experts. Strategies should be specific, but "intermediate in generality," neither too narrow nor too broad.

Example: A strategy for learning math facts efficiently is to use number families, which look like this:

4 5
——→ 9

Four facts are represented in this family: 4+5, 5+4, 9-5, 9-4. Students can use the same basic number family strategy to solve more difficult-sounding "algebraic" number puzzle problems. The strategy is to determine whether a "small" number is missing (requiring subtraction) or a "large" number (requiring addition). In

31 □
——→ 46, a small number is missing, so students subtract. The same basic strategy can be applied further to verbal problem-solving: *Juan had 16 more stamps than Frank had. Juan had 28 stamps. How many did Frank have?* Here is a map of this problem:

16 Frank
——→ Juan 28

Students create such maps by using specific steps. First, draw a number family arrow. Next, determine the large and small numbers that are given and place them on the map. Determine the operation by seeing whether the missing number is large or small. Complete the computation.

Rationale: The fundamental rationale for good strategy instruction is that it helps to ensure that all students "get it."

Implications for Materials Evaluation: Depending upon the currently popular *Zeit Geist* in education, instructional programs might not utilize overt strategies at all. Overt strategies provide an option to other, more covert and discovery oriented activities, as a means of accommodating those learners for whom discovery is ineffective or inefficient or both. Therefore, materials intended for use by learners with a broad diversity should provide conspicuous, explicit, overt strategies, as least as an easily implemented option.

Once evaluators have identified conspicuous strategies in materials, the next step is to determine whether those strategies are too broad or too narrow.

Conspicuous strategies, however, are not all created equal. A narrow "strategy" is rote-like and reflects little understanding: *invert and multiply*. A broad "strategy" only works sometimes for some students. For instance, the strategy of "drawing a picture" to solve verbal problems is insufficiently specific to consistently allow most students to solve many kinds of problems.

Finally, strategies that make sense to people who already know the objective of the strategy may not be entirely clear for those who don't: students. Evaluators should take care to examine strategies from the point of view of those who truly don't already know them.

Considerations For Adopting Mathematics Instructional Materials

Mediated Scaffolding

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Imagine that you wanted to help a toddler play on a slide. It is not likely that you would just show that student how to climb the ladder and slide down, and then say, "You're on your own now." More likely, you would help that student as much as necessary, gradually withdrawing your help as the child became more confident and competent. Scaffolding is that same type of assistance given to students in academic areas between the introduction of new knowledge and the eventual self-directed application of that knowledge.

Example: Difficult and complex strategies can be scaffolded with a series of tasks that gradually result in self-directed application. The following three tasks illustrate scaffolded steps for adding (or subtracting) fractions with unlike denominators:

Task 1

$$\frac{1}{4} = \frac{\_\_}{12}$$
$$\frac{2}{3} = \frac{\_\_}{12}$$
$$\overline{\qquad\qquad}$$
$$\frac{\_\_}{12}$$

Task 2

$$\frac{1}{3} = \_\_$$
$$+\ \frac{4}{5} = \_\_$$
$$\overline{\qquad\qquad}$$
$$\_\_$$

Task 3

$$\frac{3}{7}$$
$$+\ \frac{5}{3}$$
$$\overline{\qquad}$$

Rationale: Many children–if not most--simply will not learn and understand a complex strategy *without* some form of scaffolding. Some scaffolding can be in written form, as part of mathematics instructional materials, illustrated in the tasks above. Some scaffolding can be provided directly by teachers. For instance, instead of the written scaffolding in Task 2 above, a teacher could present Task 3 with the wording: "Remember to rewrite the fractions so that they have the same denominator. Finally, some can be provided by other students, through cooperative learning activities. However, an instructional program should facilitate scaffolding by providing sequences of tasks like those above.

Implications for Materials Evaluation: Some materials supply supplemental scaffolded tasks as part of a materials package. Such tasks would be appropriate for any student whose performance on independent tasks indicates that more work is needed before successful independent performance can be obtained.

A scaffolded sequence of tasks for every topic (or every important topic) taught in a program is the most effective means of giving students support. Task #3 above is a typical outcome task in many programs. Each of the previous, scaffolded tasks derives from that outcome task. Evaluators should identify all the tasks associated with any given Big Idea, then examine those tasks to determine whether they are scaffolded: i.e, they generally move from simpler and more contrived to more complex and natural.

Considerations For Adopting Mathematics Instructional Materials

Strategic Integration

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Integrated knowledge is the opposite of segmented knowledge. Knowledge is segmented when a given topic is taught, presumed learned, and then "dropped"; and a new topic is taught in the same fashion. One method of integrating knowledge is to combine different topics during review (See Review guideline). However, integration is also achieved through example selection during initial instruction and scaffolded instruction.

Example: Many common student errors occur as a result of a program's failure to integrate knowledge. For instance, a program might introduce these initial teaching examples for greatest common factor:

4/12 9/3 8/16 9/27 3/12 10/5

From this set, some students are likely to make the completely reasonable--but incorrect--assumption that the greatest common factor in a fraction is always one of the numbers in that fraction. An initial set of examples that preempts such confusion is: 12/30 4/12 12/18 9/27 6/8 9/12 16/8. In only three of these fractions is one number the largest common factor.

Other examples of knowledge that should be integrated:

- Subtraction problems with and without renaming
- Division with one-digit and two-digit divisors (and more as they are taught)
- Addition/subtraction and multiplication/division of fractions

Probably the most important type of integration is with various types of verbal problems.

Rationale: The principal importance of knowledge integration is that it promotes *understanding*. Students need to learn not only the "what" and "how" of mathematics, but the "when" as well. Many students, for instance, add the numerators and denominators of fractions addition problems. They haven't learned *when* they can operate on both elements of a fraction and when they cannot.

Implications for Materials Evaluation: One way in which integration is incorporated into materials is through cumulative review activities, in which all topics taught to-date are included. Such activities can be supplements to existing material, but are probably more effective as a built-in feature of materials. Short of cumulatively reviewing *all* topics taught, integration review should at least include a mixture of commonly confused topics.

A common frustration of school administrators, teachers, and others is student performance on standardized tests. Such tests can be characterized as large-scale cumulative reviews. That is, students must discriminate among all the items tested to perform successfully, which is also the way students use mathematics knowledge in the real world. Although there are valid criticisms of such tests, it seems probable that students would uniformly perform better on them if they didn't have to wait for a standardized test for the opportunity to integrate their knowledge.

Evaluators should examine materials to determine the extent to which they integrate important, potentially confusing topics (such as adding and multiplying fractions, identified above under "Examples").

Considerations For Adopting Mathematics Instructional Materials

Primed Background Knowledge

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Nearly every mathematics topic is taught with the assumption that students will build their new knowledge upon a base of existing, well-established prior knowledge. Instructional material accommodates this rather intuitive fact *in some way*, but frequently, required component knowledge is not *well-established*. An inadequate accommodation of prior knowledge may be a major cause of student difficulty with mathematics.

Example: There are common *inadequate* ways of accommodating prior, component knowledge: (1) simply "covering" requisite prior knowledge, not teaching it to a high level of mastery (see Big Ideas guideline), (2) teaching component knowledge adequately, but with a substantial gap between that instruction and the instruction on the new knowledge, and (3) introducing requisite component knowledge almost simultaneously with the new knowledge for which it is required. (See Rationale, below.) An *appropriate* schedule for teaching new knowledge and its components might look like this:

Week 1	Week 2	Week 3	Week 4	Week 5
Introduce Conspicuous Strategy for Component	"Heavy" practice on component	Begin "light" component practice	Continue light practice	Introduce the new knowledge
Scaffold Component				
Lowest Common Multiple	....	. . . .	. . . .	Adding & subtracting fractions with unlike denominators

Rationale: Students must have *well-established* knowledge of lowest common multiples (LCM) as a prerequisite to adding and subtracting fractions with unlike denominators. If LCM is covered, but not taught thoroughly, then many students are likely to have difficulties adding and subtracting fractions. If LCM is taught thoroughly, but then "dropped" for several lessons before adding and subtracting fractions is introduced, many students will have lost their "automatic" facility with LCM. And finally, if LCM is introduced just prior to adding and subtracting fractions, there is no time for students to develop *well-established* knowledge of LCM.

Implications for Materials Evaluation: Evaluators should first determine what accommodations for prior knowledge are made in a program. One major accommodation of prior knowledge is a placement tool. The other is the sequencing of topics in materials. Determine whether topics are ordered so that requisite knowledge is taught prior to the introduction of new knowledge, without either too large a gap between requisite and new knowledge or no gap at all. Something taught "last year" or even "last month" is too large a gap, and prior knowledge introduced on the same day as the new knowledge is too narrow a gap.

Considerations For Adopting Mathematics Instructional Materials

Judicious Review

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Review should be (1) sufficient, (2) distributed, (3) cumulative, and (4) varied.

Example: Here is an appropriate review schedule for a topic that has been introduced and has gone through a period of scaffolded instruction that ends on Lesson 22. (See scaffolding guideline). At first, review occurs in every other lesson.

Lesson 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57
R R R R R R R R R Etc.

Because the review in the example is *distributed*, the quantity of review *per lesson* does not have to be great: perhaps only two or three review opportunities per lesson. The distribution in the example in called an "increasing ratio" distribution—the number of lessons between reviews gradually increases.

Examples should be variations of what was taught. For instance, if students are taught a strategy for solving verbal problems in which the verb indicates the operation required, then *different* problems of that same type should be reviewed. The example below might be the first using the verb *gave*.

Dan had twenty marbles. He *gave* seven to Jill. How many does he have now?

Eventually, examples of one type should be reviewed with similar examples of different types (cumulative review). Problems in which the verb suggests an *opposite* operation, for instance, should be reviewed with those in which the verb suggests the operation.

Dan had twenty marbles. Jill *gave* him seven. How many does he have now?

Rationale: The *distribution* of practice (1) promotes retention and automaticity, and (2) helps promote transfer by requiring application of the content at different times and in different settings. *Variation* promotes transfer, but variation that is too far from what was taught does not; variation that is too great represents new learning, not review. Making review *cumulative* (1) helps prevent confusion, (2) promotes learn when to use knowledge, and (3) promotes the integration of knowledge. (See Integration Guideline).

Implications for Materials Evaluation. Evaluators can examine materials for evidence that any of the following four characteristics of effective review are present.

1. Sufficient. This judgment is subjective, but generally, more review is better than less, particularly for diverse learners, and with respect to Big Ideas for all students.
2. Distributed. Review can be distributed relatively easily by attending to some sort of distributed review schedule when creating "more review." A more sophisticated version of distributed review is an "increasing ratio" review schedule, such as that illustrated above.
3. Cumulative. (See Integration Guideline.)
4. *Appropriately* Varied. With a few unavoidable exceptions, such as basic facts, the same problems should not appear over and over again in review. (That would promote rote learning.) The more common problem, however is that problems vary *too much* : some review problems are of a different type than those taught. A strategy for subtracting without regrouping, for example, is not likely to work for problems that do require regrouping.